In-Class Exam #2 Review Sheet, Covers 7.8, 11.1-11.4
Math 280, Vanden Eynden

1. $\int_1^\infty \left(\frac{1}{e^{2x}} - \frac{2}{x^3}\right) dx$ 2. $\int_1^\infty \frac{\ln x}{x} dx$ 3. $\int_2^3 \frac{1}{\sqrt{3-x}} dx$

4. Find a formula for the general term a_n of the given sequence $\left\{\frac{1}{2}, -\frac{1}{3}, \frac{2}{9}, -\frac{4}{27}, \frac{8}{81}, \ldots\right\}$.
 Does this sequence converge or diverge? If the sequence converges, what does it converge to?

Determine whether the sequence converges or diverges. If it converges, find the limit.
Name any relevant theorems, facts and mathematical reasoning you used to reach your conclusion.

5. $a_n = e^{-n} n^4$ 6. $a_n = \left(\frac{5+n}{n}\right)^n$ 7. $a_n = \cos(\pi n)$

8. $a_n = \frac{2+n^3}{4-3n^3}$ 9. $a_n = \frac{\sin(\sqrt{n})}{\ln(n+1)}$ 10. $\left\{\frac{16}{5}, \frac{64}{25}, \frac{256}{125}, \frac{1024}{625}, \frac{4096}{3125}, \ldots\right\}$

11. Determine whether the sequence $a_n = 2 - \frac{1}{n}$ is monotonic. If it is, state whether it is increasing or decreasing. Also determine whether the sequence is bounded. If so, state the values M and m such that $m \le a_n \le M$. Is $a_n = 2 - \frac{1}{n}$ convergent?

Determine whether the series converges or diverges. If it converges, find the sum.
Name any relevant theorems, facts and mathematical reasoning you used to reach your conclusion.

12. $\sum_{n=1}^{\infty} \frac{(\sin 3)^{n-1}}{4^n}$ 13. $3 - \pi + \frac{\pi^2}{3} - \frac{\pi^3}{9} + \frac{\pi^4}{27} - \ldots$ 14. $\sum_{n=1}^{\infty} \left(\frac{1+5n^2}{n^2}\right)$ 15. $\sum_{n=1}^{\infty} \left(\frac{2n-1}{3n+4}\right)$

16. Rewrite the following series as a telescoping series. $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ Then find a formula for the nth term of the sequence of partial sums, S_n. Then evaluate $\lim_{n\to\infty} S_n$ to determine the value of the series (or state that the series diverges)

17. First verify that the terms of the sequence $\left\{ \frac{2}{\sqrt{2n-5}} \right\}_{n=3}^{\infty}$ are positive and decreasing. Then use the Integral Test to determine whether the series converges or diverges: $\sum_{n=3}^{\infty} \frac{2}{\sqrt{2n-5}}$

18. Find S_6 for $\sum_{n=1}^{\infty} \frac{1}{n^6}$. Next, find the upper bound for the remainder R_6. Finally, find lower and upper bounds on the exact value of the series, *s*.

19. Use the Comparison Test to determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

20. Use the Comparison Test to determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2+\sin n}{10^n}$

21. Use the Limit Comparison Test to determine whether the series converges or diverges.
$\sum_{n=1}^{\infty} \frac{n+3}{n^4+7n^2-n+4}$

22. Use the Limit Comparison Test to determine whether the series converges or diverges.
$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

23. Show that the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ satisfies the 3 conditions of the Integral Test. [Note: this should include an analysis of the derivative to **prove** that the terms are decreasing.] Then use the Integral Test to determine whether the series converges or diverges.

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SEQUENCES

Theorem 3:

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when *n* is an integer, then $\lim_{n\to\infty} a_n = L$.

Squeeze Theorem for Sequences:

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Theorem 6:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 7:

If $\lim_{n\to\infty} a_n = L$ and the function *f* is continuous at *L*, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Geometric sequences:

The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of *r*.

$$\lim_{n\to\infty}\{r^n\} = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Monotonic Sequence Theorem:

Every bounded, monotonic sequence is convergent.

(A) Definition Partial Sums:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum,

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence $\{s_n\}$, the sequence of partial sums $\{s_1, s_2, s_3, \ldots\}$ is convergent and its limit is a real number, $\lim_{n\to\infty} s_n = s$, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. Otherwise, the series is called divergent.

(B) Geometric Series: The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the series is divergent.

(C) Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(D) Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(E) P-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(F) Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then

$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

(G) Series Sum Estimate for the Integral Test: $s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx$

The midpoint of this interval is an estimate of *s*, with error < (half the interval's length).

(H) The Comparison Test: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

(I) The Limit Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where *c* is a finite number and $c > 0$, then either both series converge or both diverge.

ANSWERS to In-Class Exam #2 Review Sheet, Covers 7.8, 11.1-11.4
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1. $\frac{1}{2e^2}-1$, integral converges
2. Integral diverges
3. 2, integral converges
4. $a_n=\frac{1}{2}\left(\frac{-2}{3}\right)^{n-1}$, converges to 0 (geometric sequence with |r|=|-2/3|<1)
5. converges to 0 (use L'Hopital's)
6. converges to e^5 (take logarithm to evaluate limit)
7. diverges (list out several terms)
8. converges to -1/3 (divide all terms by n^3)
9. converges to 0 (use squeeze thrm)
10. converges to 0 (geometric sequence with r=4/5<1)
11. monotonic increasing, bounded by M=2. Converges to 2.
12. converges to $1/(4-\sin 3)$ (geometric series with a=1/4, r=(sin3)/4<1)
13. diverges (geometric series with |r| =|-π/3|>1)
14. diverges (sequence converges to 5 ≠ 0, Divergence Test)
15. diverges (sequence converges to 2/3 ≠ 0, Divergence Test)
16. $\sum_{n=1}^{\infty}\frac{2}{n(n+2)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$, $s_n=1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}$, $\lim_{n\to\infty}s_n=\frac{3}{2}$ (converges)
17. diverges
$$\int_3^{\infty}\frac{2}{\sqrt{2x-5}}\,dx=\lim_{t\to\infty}\int_3^{t}\frac{2}{\sqrt{2x-5}}\,dx=\lim_{t\to\infty}\left[2\sqrt{2x-5}\right]_3^t=\lim_{t\to\infty}\left[2\sqrt{2t-5}-2\right]=\infty$$
18. s_6 = 1.017326. error = $R_6\le 0.00002572$.
$1.017338\le s\le 1.017352$, so $s\approx 1.017345$ with error < 0.000007
19. converges. Use $\frac{3}{n^2+5}<\frac{3}{n^2}=3\left(\frac{1}{n^2}\right)$
20. converges. Use $\frac{2+\sin n}{10^n}\le\frac{3}{10^n}=3\left(\frac{1}{10}\right)^n$
21. converges. Use $b_n=\frac{n}{n^4}=\frac{1}{n^3}$
22. diverges. Use $b_n=\frac{1}{\sqrt{n}}$
23. converges.
$$\int_1^{\infty}\frac{x}{e^x}dx=\lim_{t\to\infty}\int_1^{t}\frac{x}{e^x}dx=\lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^t=\lim_{t\to\infty}\left[-te^{-t}-e^{-t}-\left(-e^{-1}-e^{-1}\right)\right]$$
$$=\lim_{t\to\infty}\left[-\frac{t}{e^t}-\frac{1}{e^t}+\frac{2}{e}\right]\overset{H}{=}\lim_{t\to\infty}\left[-\frac{1}{e^t}\right]-0+\frac{2}{e}=\frac{2}{e}$$